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Endogenous contest success functions: a mechanism design approach

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Abstract We describe optimal contest success functions (CSF) which maximize expected revenues of an administrator who allocates under informational asymmetry a source of rent among competing bidders. It is shown that in the case of independent private values rent administrator's optimal mechanism can always be implemented via some CSFs as posited by Tullock. Optimal endogenous CSFs have properties which are often assumed a priori as plausible features of rent-seeking contests; the paper therefore validates such assumptions for a broad class of contests. Various extensions or optimal CSFs are analyzed.

Keywords Rent seeking · Contest success functions · Asymmetric information · Bayesian mechanism design

JEL Classification D72 · D82 · C73

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1 Introduction

Tullock (1980) introduced input-output approach in rent-seeking analysis in the form of the contest success functions (CSF) that describes outcomes of contest where several agents bid resources to secure a source of rent. If $s_1, \ldots, s_n$ are participating agents' outlays in a rent-seeking contest, then obtained payoffs are given by CSFs $z_i(s_1, \ldots, s_n)$, $i = 1, \ldots, n$. Under the standard definition contests are of "winner takes all" nature, in which case CSFs describe expected gains of participants. Often the notion of contest is extended to include situations, common in rent seeking, where prizes are divisible (Hillman and Riley 1989; Corchón and Dahm 2010; Fey 2008), and success functions characterize shares of the prize obtained by contenders. Both versions permit the same description when agents are risk-neutral.[1]

While it is natural to assume some general properties of CSFs (e.g., they should monotonically increase in an agent's own rent-seeking outlays and monotonically decrease in outlays of other contenders), their exact forms are far from obvious. Since Tullock's seminal work, a simple fractional model

$$z_i(s_1, \ldots, s_n) = \frac{s_i}{\sum_{j=1}^{n} s_j} \tag{1}$$

(assuming unit value of the prize), or its immediate logit extension

$$z_i(s_1, \ldots, s_n) = \frac{\xi(s_i)}{\sum_{j=1}^{n} \xi(s_j)} \tag{2}$$

with some monotonically increasing function ξ, are widely used. Plausibility and analytical tractability were the main appeals of these forms, which explain their popularity in literature, but these and other CSFs obviously require more solid and rigorous foundations.

Two broad approaches were proposed to address this problem. The first is *axiomatic*, whereby a particular set of axioms provides necessary and sufficient conditions for a given class of CSFs. Skaperdas (1996) presented such axioms for logit CSFs (2); the centerpiece of his characterization is an appropriately formulated independence of irrelevant alternatives condition (for extensions and modifications see e.g., Münster 2009; Arbatckaya and Mialon 2010). Polishchuk and Savvateev (2004) observed that if a CSF is such that $z_i(s_1, \ldots, s_n) = \Phi\left(s_i, \sum_{j \neq i} s_j\right)$, then, with additional assumption $\Phi(0, a) = 0, \forall a > 0$, such CSF is identical to (1).

The second approach supplies *micro-foundations* for CSFs by deducing particular functional forms of rent-seeking outcomes from assumptions about rent-seeking mechanisms, institutions, and information available to participating agents. An example is the well-known model of the commons (Dasgupta and Heal 1979), when the value of the prize, which is in public domain, depends on the aggregate outlays of participating agents and is shared among them in proportion to their outlays:

[1] With risk aversion (Hillman and Katz 1984) this is no longer the case.

$$z_i(s_1, \ldots, s_n) = F\left(\sum_{j=1}^{n} s_j\right) \frac{s_i}{\sum_{j=1}^{n} s_j}; \tag{3}$$

this model, which implements the equal returns solution of the cooperative production problem (Moulin 1990), is a straightforward generalization of (1).[2] Another set of examples is given by auction-type contests when the prize goes to the highest bidder; additional randomness assumptions, e.g., that agents' outlays are augmented by random shocks (Hillman and Riley 1989; Jia 2008), or when agents are uncertain about how their bids are evaluated (Corchón and Dahm 2010), produce CSFs similar to (1), (2).

In the above examples, rules of the rent-seeking game are set exogenously and are not themselves a decision variable. Alternately in a growing strand of literature (see e.g., Dasgupta and Nti 1998; Epstein and Nitzan 2006, 2007; Corchón 2007; Corchón and Dahm 2009), it is assumed that rent-seeking contest rules are endogenous to a selection process which is driven by certain preferences over the contest outcome; this perspective moves rent-seeking studies into the fold of the mechanism design theory.

The mechanism design perspective in rent seeking brings about several important questions. First, if contest rules are set by an "administrator" who controls the source of rent and cares about rent allocation and/or collected revenues, will the administrator use CSFs as a mechanism to implement such rules? If the answer to the first question is affirmative, then, secondly, what properties such endogenous CSFs could have? In particular, can such CSFs have the logit form (2), as it is commonly assumed in the literature, and if so, under what conditions? If the logit functional form (2) does not obtain all the time, then what its features capturing general intuition about contest rules, still hold? Answers to these questions provide important insight and guidance in modeling rent-seeking contests, common in real-life situations, when rules are set by a party interested in contest outcomes.

The first of the above questions, to the best of our knowledge, has not yet been addressed in the literature, where it is usually assumed a priori that endogenous contest rules are described by some CSFs. It is often further assumed that such CSFs are *partially* endogenous, i.e., they belong to a certain a priori selected class (e.g., the class of logit functions (2) where ξ is concave, as in Dasgupta and Nti 1998; see also Corchón 2007, or with $\xi(s_i) = s_i^r$, as in Wang 2010) from which the resource administrator makes her selection. Upfront assumptions about CSF forms could reflect e.g., institutional restrictions on rent allocation mechanism such as requirements of competitive bidding and collusion-proofness; such assumptions, however, are not endogenous to a mechanism design process and properties of participating agents.

In some instances (as in e.g., Dasgupta and Nti 1998), partially endogenous CSFs are derived in the case of full (symmetric) information when the administrator has complete knowledge of agents' types, preferences, etc. However, in such case the very usage of CSF becomes superfluous and quite likely not in the best interests of the resource administrator. Indeed, with full information the administrator can simply

[2] For applications of this model in rent seeking analysis see e.g., Grossman (1994).

identify the first-best outcome, subject to appropriate participation constraints, and present agents with "take it or leave it" offers that would implement such outcome.[3]

If the administrator does not have full information about agents' types, top-down take-it-or-leave-it-type contracts could no longer be optimal and not even, for that matter, feasible. In such case, bidding is not a constraint imposed upon the administrator, but her instrument of choice, since bids, apart from their immediate material value, are also signals that reveal valuable information about agents' types.[4] Optimal CSFs that describe administrator's best response to such signals thus become fully endogenous.

Below we derive optimal CSFs through Bayesian implementation assuming independent private values, i.e., that agents' types—their valuations of the resource allocated by the administrator which are known only to agents themselves—are randomly drawn from a given distribution which is a common knowledge.[5] It is shown that no matter what mechanism the administrator uses to communicate with the agents, as long as it allocates the source of rent against side payments to the administrator, the best of all such mechanisms can always be represented through some CSFs. Therefore, under very general assumptions, *Bayesian implementation endogenizes the very CSF-based model of rent seeking*. Furthermore, such implementation ensures certain properties of optimal CSFs, analogous to those of Tullock's logit functions (2). If (and, for $n = 2$, only if) in addition agents have Cobb-Douglas utilities, optimal CSFs are logit, irrespective of distribution of agents' types.

To get further insight into the class of CSFs obtained through Bayesian mechanism design, we study asymptotic behavior of such functions when the number of participating agents grows to infinity. It is shown that in such case optimal CSF allow a unidimensional approximation. Asymptotic analysis also reveals increasing returns to scale in rent seeking as hypothesized in Murphy et al. (1993) and possible exclusion from rent seeking of agents with low valuation of the prize—a phenomenon observed under different assumptions by Hillman and Riley (1989). Finally, as an extension of the base model, we derive optimal CSFs when the administrator can invest a portion of agents' contributions to augment the allocated resource—in such case, still assuming Cobb-Douglas utilities, optimal CSFs combine features of the forms (2) and (3).

The rest of the paper is organized as follows. In Sect. 2, a Bayesian mechanism design problem leading to optimal CSFs is presented. This problem is solved in Sect. 3, producing a class of endogenous CSFs. Properties of the derived CSFs are analyzed in Sect. 4, including conditions under which such CSFs admit logit representation

[3] "Ideally, a planner would simply read everyone's mind and impose the feasible outcome that maximizes his objective" (Palfrey and Srivastava 1993, p. 3). Rent-seeking equilibria based on commonly used CSFs usually leave agents above their reservation utility levels, despite of partial dissipation of rent; this is an indication that such equilibria are not first-best outcomes for the rent administrator (assuming that the administrator's sole concern is revenue collection and that she does not care about agents' welfare—for a more general formulation see Epstein and Nitzan 2006).

[4] Other types of informational asymmetry are considered in the rent-seeking literature as well; e.g., in Wärneryd (2009) it is assumed that agents are to various degrees informed about the value of the prize.

[5] There could be other kinds of private information, e.g., costs of rent-seeking outlays (efforts) to agents, as in Fey (2008); however, such case could be re-formulated in terms of unobservable valuations. Malueg and Yates (2004) study Bayesian equilibrium in rent-seeking contests when agents' types are statistically dependent.

proposed by Tullock. In the same section, we also provide examples of endogenous CSFs. Section 5 investigates asymptotic properties of optimal CSFs for large numbers of participating agents. In Sect. 6, the analysis is extended on rent-seeking contests when the source rent is of variable size and can be enhanced by investing some of the payments collected from rent-seekers. Section 7 concludes.

2 Rent seeking and Bayesian mechanism design

Consider a model where the administrator allocates one unit of resource among n rent-seeking agents (we assume that the resource has no direct value to the administrator). Each agent has a quasi-linear utility function $w_i\varphi(z_i) - s_i,\ i = 1, \ldots, n$; here z_i is the quantity of resource obtained from the administrator, s_i—rent-seeking outlay, and w_i—agent's type which is his private information. The function $\varphi(\cdot)$ is smooth for $z > 0$ and satisfies the following conditions: $\varphi'(z) > 0$, $\varphi''(z) < 0$ and $\varphi(0) = 0$. Unless explicitly stated otherwise, we will also be assuming that $\lim_{z\to 0}\varphi'(z) = \infty$. One way to interpret agents' types is to view them as endowments of another resource, complementary to the one allocated by the administrator.[6]

Agents' types are randomly and independently drawn from a distribution with cumulative function $G(w)$ and density $g(w)$, $w \in [\underline{w}, \overline{w}]$, $0 \le \underline{w} < \overline{w} \le \infty$; this distribution is common knowledge to all parties involved. The function $\rho(w) \equiv w - \frac{1-G(w)}{g(w)}$ (marginal revenue, or valuation, as it is known in the auction theory—see e.g., Klemperer 1999) is assumed monotonically increasing—a condition which is satisfied for most commonly used distributions, including those with increasing hazard rate $\frac{g(w)}{1-G(w)}$. Both the administrator and agents are risk-neutral.

Informational asymmetry prompts the administrator to communicate with agents prior to allocating the resource by using a mechanism $\mathcal{M} = \langle M_1, \ldots, M_n; a(\cdot)\rangle$ which is a collection of strategy sets from which agents select their messages $m_i \in M_i$, and an allocation function $a(m_1, \ldots, m_n)$ which describes administrator's decision in response to received messages. In mechanisms considered below $a : M_1 \times \cdots \times M_n \to \mathbb{R}^n \times \left\{z \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^n z_i \le 1\right\}$; an allocation thus comprises a set of payments $s_1, \ldots, s_n$ of the agents to the administrator and a division $\sum_{i=1}^n z_i \le 1$ of the unit stock of resource among the agents:

$$a(m_1, \ldots, m_n) = \langle s_1(m_1, \ldots, m_n), \ldots, s_n(m_1, \ldots, m_n); \\ z_1(m_1, \ldots, m_n), \ldots, z_n(m_1, \ldots, m_n)\rangle. \quad (4)$$

This mechanism works as follows: once all agents have communicated to the administrator their messages m_i, $i = 1, \ldots, n$, agent i is required to make the payment $s_i(m_1, \ldots, m_n)$ to the administrator and receives in exchange the amount $z_i(m_1, \ldots, m_n)$ of the allocated resource. Notice that there are no a priori restrictions on the content of messages (or, what is the same, on information sets); in particular, no communication is also an option with a pre-set constant allocation function.

[6] Most of our results also hold for a general constant returns to scale two-input production function $f(z, w)$.

Contest success functions form a subset of such mechanisms, whereby messages are payments $s_i \geq 0$ offered to the administrator (such promises are binding), and the allocation is the $2n$-tuple $\langle s_1, \ldots, s_n;\ z_1(s_1, \ldots, s_n), \ldots, z_n(s_1, \ldots, s_n)\rangle$ of rent-seeking outlays and outcomes. It will be shown in the next section that the administrator can restrict her choice of the best mechanism to this subset.

We assume Bayesian mechanism implementation, in which case agents' strategies form a Bayes-Nash equilibrium—they are functions $m_i : [\underline{w}, \overline{w}] \to M_i,\ i = 1, \ldots, n$ of agents' types, such that for every type w_i $m_i(w_i)$ maximizes agent i's expected utility, conditional on other agents playing strategies $m_j\left(w_j\right),\ j \neq i$:

$$\begin{aligned} &E_{w_{-i}}\left[w_i \varphi\left(z_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right)\right) - s_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right)\right] \\ &\quad \geq E_{w_{-i}}\left[w_i \varphi\left(z_i\left(m_i', m_{-i}\left(w_{-i}\right)\right)\right) - s_i\left(m_i', m_{-i}\left(w_{-i}\right)\right)\right], \quad i = 1, \ldots, n, \end{aligned} \tag{5}$$

for all feasible messages $m_i' \in M_i$.[7]

Since participation in rent-seeking game is voluntary, the administrator also needs to ensure that equilibrium outcomes leave agents (non-strictly) above their reservation utility levels which in the present context equal zero:

$$\begin{aligned} &E_{w_{-i}}\left[w_i \varphi\left(z_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right)\right) - s_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right)\right] \geq 0, \\ &\forall w_i \in \left[\underline{w}, \overline{w}\right],\ i = 1, \ldots, n. \end{aligned} \tag{6}$$

Now the optimal rent-seeking mechanism design problem can be stated as maximization of the expected gross payoff collected by the administrator from the agents

$$\max E \sum_{i=1}^{n} s_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right) \tag{7}$$

over all mechanisms (4) subject to conditions (5), (6) and the resource constraint

$$\sum_{i=1}^{n} z_i\left(m_i\left(w_i\right), m_{-i}\left(w_{-i}\right)\right) \leq 1, \quad \forall w_1, \ldots, w_n \in \left[\underline{w}, \overline{w}\right]. \tag{8}$$

3 Optimal contest success functions

The problem of optimal mechanism design is considerably simplified when mechanisms are *direct*, i.e., agents' messages are announcements (truthful or otherwise) of their types. In the present context, a direct mechanism includes strategy sets $M_i = \left[\underline{w}, \overline{w}\right]$ and functions $\tilde{s}_i$ and $\tilde{z}_i$, defined over $\underbrace{\left[\underline{w}, \overline{w}\right] \times \cdots \times \left[\underline{w}, \overline{w}\right]}_{n \text{ times}}$, $i = 1, \ldots, n$, such that once agents' types are reported as w_i', the mechanism requires agent $i =$

[7] We use the notation "$-i$" as a conventional shortcut for "all variables other than i". For more on Bayes-Nash equilibria in rent-seeking games with asymmetric information see Malueg and Yates (2004), Fey (2008).

$1, \ldots, n$ to make the payment $\tilde{s}_i\left(w_1', \ldots, w_n'\right)$ to the administrator against obtaining from her $\tilde{z}_i\left(w_1', \ldots, w_n'\right)$ units of the allocated resource. Direct mechanism is *incentive-compatible* if correct reporting by agents of their types constitutes a Bayes-Nash equilibrium, i.e.,

$$E_{w_{-i}}\left[w_i\varphi(\tilde{z}_i\left(w_i, w_{-i}\right)) - \tilde{s}_i\left(w_i, w_{-i}\right)\right] \\ \geq E_{w_{-i}}\left[w_i\varphi(\tilde{z}_i\left(w_i', w_{-i}\right)) - \tilde{s}_i\left(w_i', w_{-i}\right)\right], \quad \forall w_i, w_i' \in \left[\underline{w}, \overline{w}\right], \quad i = 1, \ldots, \tag{5'}$$

According to the revelation principle (Myerson 1981), if functions $m_1(\cdot), \ldots, m_n(\cdot)$ form a Bayes-Nash equilibrium for a mechanism (4), then the functions

$$\begin{aligned} \tilde{s}_i\left(w_1', \ldots, w_n'\right) &= s_i\left(m_1(w_1'), \ldots, m_n(w_n')\right), \\ \tilde{z}_i\left(w_1', \ldots, w_n'\right) &= z_i\left(m_1(w_1'), \ldots, m_n(w_n')\right), \end{aligned} \tag{9}$$

$i = 1, \ldots, n$, represent a direct incentive-compatible mechanism such that for any combination of agents' types $w_1, \ldots, w_n$ the two mechanisms yield the same allocation. Therefore, the choice of optimal mechanisms can be confined to direct mechanisms $\tilde{z}_i(\cdot)$, $\tilde{s}_i(\cdot)$, and the administrator's problem set forth in the previous section can be re-stated as follows:

$$\max E\left[\sum_{i=1}^{n} \tilde{s}_i\left(w_i, w_{-i}\right)\right] \tag{7'}$$

subject to the resource constraint

$$\sum_{i=1}^{n} \tilde{z}_i(w_i, w_{-i}) \leq 1, \ \forall w_1, \ldots, w_n \in \left[\underline{w}, \overline{w}\right] \tag{8'}$$

the incentive compatibility constraints (5′), and participation constraints

$$E_{w_{-i}}[w_i\varphi(\tilde{z}_i\left(w_i, w_{-i}\right)) - \tilde{s}_i\left(w_i, w_{-i}\right)] \geq 0, \quad \forall w_i \in \left[\underline{w}, \overline{w}\right], \quad i = 1, \ldots, n. \tag{6'}$$

We will now demonstrate that the optimal solution of this problem (delivering the best results over all conceivable mechanisms (4)) can be implemented by appropriately chosen CSFs. To this end, first notice that in a direct incentive-compatible mechanism $\tilde{s}_i(\cdot)$, $\tilde{z}_i(\cdot)$ satisfying participation constraints (6′) transfer functions $\tilde{s}_i\left(w_i, w_{-i}\right)$ can be replaced by $\tilde{s}_i\left(w_i\right) \equiv E_{w_{-i}}\tilde{s}_i\left(w_i, w_{-i}\right)$ (for simplicity, we keep the same notation for such reduced single-variable form). Indeed, as it follows from the definitions, such new mechanism remains incentive-compatible, also meets participation constraints, and yields the same value to the maximand (7′). Therefore, without loss of generality

$\tilde{s}_i(\cdot)$ can be assumed depending on w_i alone; this assumption is kept through the rest of the paper.

At the next stage, the problem (5′)–(8′) is solved. To this end, denote $F(t) \equiv \left(\varphi'\right)^{-1}(1/t)$ and consider a symmetric function $A_F(x_1, \ldots, x_n)$ which is uniquely determined for all $x_1 \geq 0, \ldots, x_n \geq 0$, $\sum_{i=1}^{n} x_i > 0$ by the following equation:

$$\sum_{i=1}^{n} F\left(\frac{x_i}{A_F(x_1, \ldots, x_n)}\right) = 1. \tag{10}$$

In what follows we use the notation $[x]_+ \equiv \max(x, 0)$; recall that $\rho(w)$ is the marginal revenue function for distribution G.

Proposition 1 *Optimal direct mechanism $\tilde{s}_i(\cdot)$, $\tilde{z}_i(\cdot)$, $i = 1, \ldots, n$, which solves the problem (5′)–(8′), is as follows:*

$$\tilde{z}_i(w_i, w_{-i}) = F\left(\frac{[\rho(w_i)]_+}{A_F\left([\rho(w_1)]_+, \ldots, [\rho(w_n)]_+\right)}\right), \quad i = 1, \ldots, n \tag{11}$$

(*if* $\rho(w_i) \leq 0$ *for all* i, *then all* $\tilde{z}_i$ *are equal zero*); *and*

$$\tilde{s}_i(w_i) = \tilde{s}(w_i) \equiv w_i \overline{\varphi}(w_i) - \int_{\underline{w}}^{w_i} \overline{\varphi}(s)\, \mathrm{d}s, \tag{12}$$

where

$$\overline{\varphi}(w_i) \equiv E_{w_{-i}} \varphi(\tilde{z}_i(w_i, w_{-i})). \tag{13}$$

Proofs of this and subsequent propositions can be found in the Appendix.

Finally, a set of endogenous CSFs which solve the optimal mechanism design problem (without an a priori requirement that such mechanism is CSF-based) obtains from the above direct mechanism. To this end, one has to eliminate agents' types w_i from (11), (12). Recall that the marginal revenue function ρ monotonically increases in type, and therefore there exists $w^0 \in [\underline{w}, \overline{w})$ such that $\rho(w_i) > 0$ for all $w_i \in [\underline{w}, \overline{w}]$, $w_i > w^0$, and $\rho(w_i) < 0$ for all $w_i \in [\underline{w}, \overline{w}]$, $w_i < w^0$. Notice further that for all $w_i < w^0$ agent i obtains no resource from the administrator and hence due to (12), (13) makes no contribution, whereas for $w_i > w^0$ both amounts are positive. It is shown in the Appendix that over the domain $[w^0, \overline{w}]$ the function $\tilde{s}(\cdot)$ monotonically increases, and therefore the mechanism (11)–(13) indeed yields CSFs which solve the problem (5)–(8).

Proposition 2 *The function $\tilde{s}(\cdot)$ monotonically increases in the range $s \in [\underline{s}, \overline{s}]$, where $\underline{s} = \tilde{s}\left(w^0\right)$, $\overline{s} = \tilde{s}(\overline{w})$, and optimal CSFs solving the problem* (5)–(8) *are* *defined over* $s_i \in \left[\underline{s}, \overline{s}\right]$, $i = 1, \ldots, n$ *as follows:*

$$z_i\left(s_i, s_{-i}\right) = F\left(\frac{\eta\left(s_i\right)}{A_F\left(\eta\left(s_1\right), \ldots, \eta\left(s_n\right)\right)}\right), \quad i = 1, \ldots, n, \tag{14}$$

where

$$\eta(s) \equiv \rho\left(\tilde{s}^{-1}(s)\right).[8] \tag{15}$$

The function inverse to $\eta(s)$ can be derived from a differential equation which could simplify the calculation of optimal CSFs and obtains as follows. Due to (15) $\frac{\mathrm{d}\eta}{\mathrm{d}s} = \frac{\mathrm{d}\rho}{\mathrm{d}w} \Big/ \frac{\mathrm{d}\tilde{s}}{\mathrm{d}w}$, and according to (12) and (13)

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}w} &= w\overline{\varphi}'(w) = w\frac{\partial}{\partial w}E_{w_2,\ldots,w_n}\varphi\left(F\left(\frac{\rho(w)}{A_F\left(\rho(w), \rho(w_2), \ldots, \rho(w_n)\right)}\right)\right)\\ &= w\rho'(w)E_{w_2,\ldots,w_n}\varphi'\left(F\left(\frac{\rho(w)}{A_F}\right)\right)F'\left(\frac{\rho(w)}{A_F}\right)\left(\frac{1}{A_F} - \frac{\rho(w)}{A_F^2}A_{F x_1}\right).\end{aligned}$$

By definition of F, one has $\varphi'\left(F\left(\frac{\rho(w)}{A_F}\right)\right) = \frac{A_F}{\rho(w)}$, and the previous expression simplifies to

$$\frac{\mathrm{d}s}{\mathrm{d}w} = w\rho'(w)E_{w_2,\ldots,w_n}F'\left(\frac{\rho(w)}{A_F}\right)\left(\frac{1}{\rho(w)} - (\ln A_F)_{x_1}\right).$$

Therefore,

$$\frac{\mathrm{d}s}{\mathrm{d}\eta} = w(\eta)E_{w_2,\ldots,w_n}F'\left(\frac{\eta}{A_F}\right)\left(\frac{1}{\eta} - (\ln A_F)_{x_1}\right), \tag{16}$$

where $w(\eta)$ is the function inverse to the marginal revenue function $\rho(w)$. By integrating (16) with the initial condition $\eta_0 = \rho(\underline{w})$, $s_0 = \underline{w}\overline{\varphi}(\underline{w})$, one obtains the function inverse to $\eta(s)$.

4 Properties and examples of optimal contest success functions

Note that the function F is monotonically increasing and $F(0) = 0$ due to $\varphi'(0) = \infty$. Generally, CSFs (14) are *not* of Tullock's logit form (2), although they share with that form some common properties. Thus, both classes of CSFs—(2) and (14)—conform to the basic intuition of rent-seeking technologies—rent-seeking outcome for a given agent increases in his own outlay s_i and decreases in the outlays of all other agents; furthermore, such outcome is determined by a ratio of an appropriate valuation (monotone

[8] These functions can be extrapolated beyond the "equilibrium domain" $\left[\underline{s}, \overline{s}\right]$ by letting $z_i\left(s_i, s_{-i}\right) = 0$ for $s_i < \underline{s}$ and $z_i\left(s_i, s_{-i}\right) = z_i\left(\overline{s}, s_{-i}\right)$ for $s_i > \overline{s}$.

transformation) of the agent's outlay $\eta(s_i)$ to an aggregate (average) of such valuations of outlays of all agents.

Proposition 3 *The following statements hold:*

(i) *The function $A_F(x_1, \ldots, x_n)$ is monotonically increasing in its arguments.*
(ii) *The function $t_n A_F$, where $nF(t_n) = 1$, is a generalized average of $x_1, \ldots, x_n$ in that it is symmetric, exhibits constant returns to scale, and is such that* $\min x_i \leq t_n A_F(x_1, \ldots, x_n) \leq \max x_i$*; in particular $t_n A_F(x, \ldots, x) = x$.*[9]
(iii) *The function $z_i(s_i, s_{-i})$ monotonically increases in $s_i \in [\underline{s}, \overline{s}]$ and monotonically decreases in $s_j \in [\underline{s}, \overline{s}]$ for all $j \neq i$.*

CSF (14) can be reduced to the logit form (2) if the utility function is of Cobb-Douglas type: $\varphi(z) = \alpha^{-1} z^{\alpha}$, $\alpha \in (0, 1)$. In such case $F(t) = t^{\frac{1}{1-\alpha}}$, $A_F(x_1, \ldots, x_n) = \left(\sum_{i=1}^n x_i^{\frac{1}{1-\alpha}}\right)^{1-\alpha}$, $t_n = n^{-(1-\alpha)}$, and $t_n A_F(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^n x_i^{\frac{1}{1-\alpha}}\right)^{1-\alpha}$ is the generalized (power) mean of $x_1, \ldots, x_n$ with exponent $1/(1-\alpha)$. In this case, CSFs (14) take the following logit form:

$$z_i(s_i, s_{-i}) = \frac{(\eta(s_i))^{\frac{1}{1-\alpha}}}{\sum_{j=1}^n \left(\eta(s_j)\right)^{\frac{1}{1-\alpha}}}, \quad i = 1, \ldots, n. \tag{17}$$

It turns out that Cobb-Douglas utility is not just sufficient, but, for $n = 2$, also necessary for logit representation of CSF (17). To make this statement precise, call the range $[\underline{z}, \overline{z}]$, $\underline{z} \equiv z_i(\underline{s}, \overline{s}, \ldots, \overline{s})$, $\overline{z} \equiv z_i(\overline{s}, \underline{s}, \ldots, \underline{s})$, the effective domain of CSF (14).

Proposition 4 *Contest success functions* (14) *admit logit representation* (2) *if and, in case $n > 2$, only if in the effective domain $[\underline{z}, \overline{z}]$ one has $\varphi(z) = B_0 + B_1 z^{\alpha}$, with some $\alpha \in (0, 1)$, $B_1 > 0$; $B_0 = 0$ when $\underline{z} = 0$ and $B_0 + (1-\alpha)\underline{z}^{\alpha} B_1 > 0$ when $\underline{z} > 0$.*[10]

The following example illustrates the derivation of logit CSF for a Cobb-Douglas utility function. Let $n = 2$, $\varphi(z) = 2\sqrt{z}$, and the distribution of w is of Pareto type: $\underline{w} = 1$, $\overline{w} = \infty$, $G(w) = 1 - 1/w^2$, $g(w) = 2/w^3$, $\rho(w) = w/2$.[11] In this case

[9] For a similar but more restrictive concept of generalized averages, see Kolmogorov (1985). For more on the role of homogeneity in CSF, see Malueg and Yates (2006).

[10] When $n = 2$, the logit form obtains for a class of utility functions broader than Cobb-Dougals's—one can show that in this case utility functions generating logit CSFs are such that $\varphi'(z) = \theta(z) z^{\alpha-1}$, $\alpha \in (0, 1)$, where the function $\theta(\cdot)$ is symmetric around 1/2: $\theta(z) = \theta(1-z)$, $\forall z \in [0, 1]$, and $\varphi'(z)$ remains monotonically decreasing. However, for $n = 2$ every optimal CSF (14) admits a generalized difference-form representation introduced under different assumptions by Dixit (1987). Indeed, since A_F exhibits constant returns to scale, one has $z_1(s_1, s_2) = F\left(\frac{1}{A_F(1, \eta(s_2)/\eta(s_1))}\right) = H(\ln \eta(s_1) - \ln \eta(s_2))$, for some monotonously increasing function H.

[11] Pareto distribution is a plausible assumption when types are interpreted as endowments of a complementary resource.

$F(t) = t^2,\ A_F(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ and the differential equation (16) is as follows:

$$\frac{ds}{d\eta} = 2w(\eta)E_{w_2}\left[\frac{1}{A_F} - \frac{\eta A_{F x_1}}{A_F^2}\right] = 4\eta \int_1^\infty \frac{w_2^2/4}{\left(\eta^2 + w_2^2/4\right)^{3/2}} \cdot \frac{2}{w_2^3}\, dw_2$$
$$= \frac{2}{\eta^2} \ln\left(\sqrt{1 + 4\eta^2} + 2\eta\right) - \frac{4}{\eta\sqrt{1 + 4\eta^2}}.$$

Integration of the above equation yields

$$s(\eta) = -\frac{2}{\eta}\ln\left(\sqrt{1 + 4\eta^2} + 2\eta\right) + C,$$

and due to the initial condition $\eta_0 = \rho(\underline{w}) = 1/2,\ s_0 = \overline{\varphi}(1) = 2\sqrt{2} - 2\ln\left(1 + \sqrt{2}\right)$, one obtains

$$s(\eta) = 2\ln\left(1 + \sqrt{2}\right) + 2\sqrt{2} - \frac{2}{\eta}\ln\left(\sqrt{1 + 4\eta^2} + 2\eta\right).$$

Figure 1 illustrates the function $\xi(s) = (\eta(s))^2$ which enters the corresponding logit CSF (2).

Utility functions other than Cobb-Douglas generate various CSFs, some of which are also known from the literature.[12] Thus, for $\varphi(z) = 1 - \mathrm{e}^{-z}$ one has $F(t) = \ln t$ and $A_F(x) = \left(\prod_{i=1}^n x_i/\mathrm{e}\right)^{1/n}$; in this case, $t_n = \mathrm{e}^{1/n}$ and $t_n A_F(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is the geometric mean of $x_1, \ldots, x_n$. The corresponding CSF is of the form

$$z_i(s_i, s_{-i}) = \ln\eta(s_i) - \frac{1}{n}\sum_{j=1}^n \ln\eta(s_j) + \frac{1}{n}, \quad i = 1, \ldots, n. \tag{18}$$

To illustrate such CSFs, again assume $n = 2$, in which case (18) is reduced to

$$z_i(s_i, s_{-i}) = \frac{1}{2}\left(\ln\eta(s_i) - \ln\eta(s_{-i})\right) + \frac{1}{2}, \quad i = 1, 2, \tag{19}$$

and the equation (16) is as follows:

$$\frac{\mathrm{d}s}{\mathrm{d}\eta} = \frac{w(\eta)}{2\sqrt{\mathrm{e}\eta^3}} E_{w_2}\sqrt{\rho(w_2)}.$$

Consider a truncated Pareto distribution over the range $\underline{w} = 1,\ \overline{w} = \sqrt{\mathrm{e}}$, with $G(w) = \frac{w-1}{\sqrt{\mathrm{e}}-1} \cdot \frac{\sqrt{\mathrm{e}}}{w},\ g(w) = \frac{\sqrt{\mathrm{e}}}{\sqrt{\mathrm{e}}-1} \cdot \frac{1}{w^2},\ \rho(w) = \frac{w^2}{\sqrt{\mathrm{e}}}$. Integration of the above equation for

[12] In the next two examples $\varphi'(0)$ is finite, but the preceding theory remains applicable for as long as agents' types do not differ from each other too much.

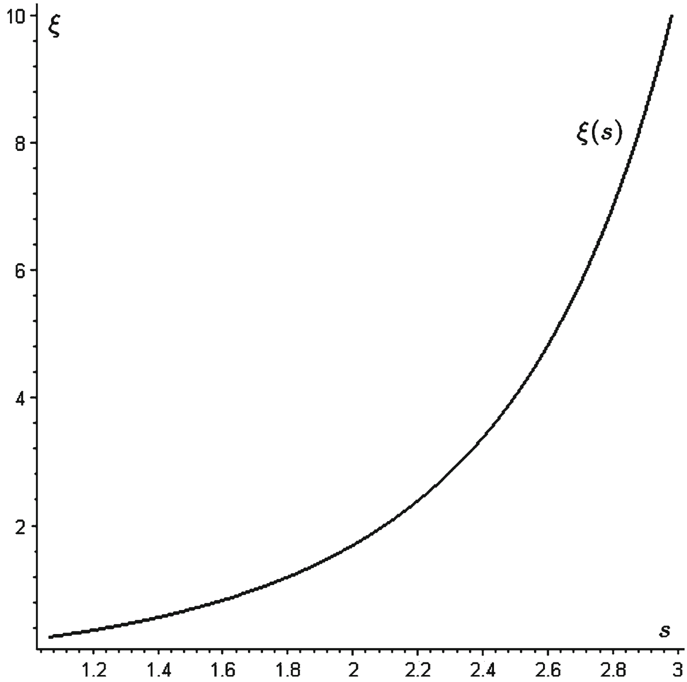

Fig. 1 Example of logit CSF

this distribution leads to

$$s\,(\eta) = s_0 + \frac{1}{4\left(\sqrt{e}-1\right)}\,(\ln\eta - \ln\eta_0),$$

and hence the CSF (19) takes the following form

$$z_i\,(s_i, s_{-i}) = 2\left(\sqrt{\mathrm{e}}-1\right)(s_i - s_{-i}) + \frac{1}{2}, \quad i = 1, 2. \tag{20}$$

CSFs (19) and (20) are examples of difference-form success functions considered e.g., by Baik (1998) and Che and Gale (2000).[13]

In a yet another example $\varphi\,(z) = \ln(z+1)$, in which case $F\,(t) = t - 1,\ A_F\,(x) = \frac{1}{n+1}\sum_{i=1}^{n} x_i,\ t_n = \frac{n+1}{n}$ and $t_n A_F\,(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the conventional arithmetic

[13] One can verify that for a general distribution $G(w)$ the expressions (19) hold as long as $\frac{1}{\mathrm{e}} \le \frac{\eta(s_i)}{\eta(s_{-i})} \le \mathrm{e}$, which in its turn require that the variation of agents' types is not too large. Otherwise for the optimal CSF these expressions should be modified as follows:

$$z_i\,(s_i, s_{-i}) = \max\left\{\min\left\{\frac{1}{2}\left(\ln\eta(s_i) - \ln\eta(s_{-i})\right) + \frac{1}{2}, 1\right\}, 0\right\}, \quad i = 1, 2,$$

similarly to Che and Gale (2000).

mean of $x_1, \ldots, x_n$. The corresponding CSFs

$$z_i\left(s_i, s_{-i}\right) = \frac{(n+1)\,\eta\left(s_i\right) - \sum_{j=1}^{n} \eta(s_j)}{\sum_{j=1}^{n} \eta(s_j)} \tag{21}$$

combine features of logit and difference forms, and the curve $\eta\,(s)$ can be obtained from the differential equation

$$\frac{\mathrm{d}s}{\mathrm{d}\eta} = \frac{w\,(\eta)}{\eta} - w\,(\eta)\,E_{w_2,\ldots,w_n}(\ln A_F)_{x_1}.$$

The resource allocation mechanism which is implemented by the optimal CSF is by definition interim (*after* the agents have learned their types but *before* those are communicated to the resource administrator) efficient among all incentive-compatible mechanism: it maximizes the expected revenues of the administrator while agents' expected utilities remain above their zero reservation levels. In general, however, due to the gap between agents' actual and marginal valuations (resp. w and $\rho(w)$) this mechanism is not efficient ex post—the only exception is the Pareto distribution $G\,(w) = 1 - (\underline{w}/w)^{\,k},\ k > 1,\ \underline{w} > 0,\ \overline{w} = \infty$, when $\rho\,(w)/w = \text{const}$.[14] Efficiency losses are the toll of informational asymmetry; such losses are especially severe for "low" types w_i, and in the case $\rho\left(\underline{w}\right) < 0$ (or, what is the same, $\underline{w} < w^0 < \overline{w}$) take the extreme form of complete exclusion of agents in the $\left[\underline{w}, w^0\right]$ range from the resource allocation process, whereas social efficiency requires allocation of positive amounts of the resource to *all* agents with $w_i > 0$.[15] If agents' types are treated, as in Sect. 2, as endowments of a complementary production input, such exclusion could be interpreted as *informational discrimination* of poorer agents, which are "too small" to be of interest for the resource administrator, and their participation would have restricted her ability to extract revenue from wealthier rent-seekers.[16] This observation sheds new light on the causes of entry barriers that owners of small assets face: in addition to political economy/public choice explanations (Djankov et al. 2002; Polishchuk 2008) and inequality of stakes arguments (Hillman and Riley 1989), such discrimination could also have informational rationales.

The above described efficiency losses are in fact avoidable if the mechanism design objective is to maximize (ex post) social welfare, rather than expected revenue of the resource administrator. In general, ex post efficiency requirement could be inconsistent with incentive compatibility, balanced budget, and participation constraints (see e.g., Myerson and Satterthwaite 1983). It is noteworthy that in the present case such requirements can be reconciled; however, if mechanism design is in the hands of

[14] In the case of non-divisible resource of no direct value to the seller, optimal auctions with symmetric bidders always allocate the resource to a bidder with the highest valuation, as long as the marginal valuation function monotonically increases and $\rho(\underline{w}) > 0$ (Klemperer 1999).

[15] Social losses and rent dissipation due to informational asymmetry in rent-seeking contest were observed in a different setting in Hillman and Riley (1989).

[16] Similarly a price-discriminating monopolist could elect under informational asymmetry not to cater to lower wealth/valuation segment of the market in order to enhance the yield of the more lucrative part.

the resource administrator, ex post efficient mechanisms, while available, will not be selected for distributions other than Pareto.

Proposition 5 *There exists a direct ex post efficient mechanism* $\tilde{s}_i(\cdot), \tilde{z}_i(\cdot)$, $i = 1, \ldots, n$, *which is incentive-compatible, satisfies the resource constraint* (8′)*, and meets the participation constraints* (6′) *for rent-seeking agents and a similar constraint for the resource administrator:*

$$E\left[\sum_{i=1}^{n} \tilde{s}_i(w)\right] \geq 0. \tag{22}$$

Moreover, such mechanism can be selected among the following Groves mechanisms

$$(\tilde{z}_1(w), \ldots, \tilde{z}_n(w)) = \arg\max\left\{\sum_{i=1}^{n} w_i \varphi(z_i) \mid \sum_{i=1}^{n} z_i \leq 1\right\},$$
$$\tilde{s}_i(w) = -\sum_{j \neq i} w_j \varphi(\tilde{z}_j(w)) + k_i, \quad i = 1, \ldots, n \tag{23}$$

with some $k_1, \ldots, k_n =$ const.

Notice finally that the direct mechanism (11)–(13) which underlies the optimal CSF (14) could have multiple Bayes-Nash equilibria. The truth-telling equilibrium which is one of them implements the allocation desired by the resource administrator, but others might not do so. To eliminate such undesirable additional equilibria, if they should occur, the initial mechanism could be modified to make it *fully implementing*, in which case *all* equilibria under the new mechanism entail the desired allocation. Full Bayesian implementation is possible when an appropriately formulated *monotonicity condition* is satisfied (see e.g., Palfrey 1992). It is shown by d'Aspremont et al. (2005) that for a class of auction-type mechanisms with independent distributions of agents' types (these assumptions are met in our case) for every incentive-compatible equilibrium there is a mechanism which implements such equilibrium *essentially uniquely* in the following sense: in all the equilibria of such mechanism the expected revenues of the resource administrator are the same as in the initial incentive-compatible equilibrium.

5 Limiting case: a continuous model

Additional insight into properties of endogenous CSFs can be gained by considering the limiting case of an "atomless" model which approximates rent seeking with a large number of participants.

Suppose that rent-seekers form a unit continuum of agents with the distribution $G(w)$ of their types, and the resource administrator allocates one unit of resource across this continuum by implementing a direct mechanism $\tilde{s}_\infty(\cdot)$, $\tilde{z}_\infty(\cdot)$, so that an agent that reveals his type as w gets $\tilde{z}_\infty(w)$ units of resource against a contribution of

$\tilde{s}_\infty(w)$.[17] This mechanism is incentive-compatible iff

$$w\varphi\left(\tilde{z}_\infty(w)\right) - \tilde{s}_\infty(w) \geq w\varphi\left(\tilde{z}_\infty\left(w'\right)\right) - \tilde{s}_\infty\left(w'\right), \quad \forall w, w' \in [\underline{w}, \overline{w}], \tag{24}$$

and the participation constraint takes form

$$w\varphi\left(\tilde{z}_\infty(w)\right) - \tilde{s}_\infty(w) \geq 0, \quad \forall w \in [\underline{w}, \overline{w}]. \tag{25}$$

The optimal mechanism maximizes the administrator's aggregate revenues $\int_{\underline{w}}^{\overline{w}} \tilde{s}_\infty(w)\, g(w)\, \mathrm{d}w$ subject to constraints (24), (25) and the resource constraint $\int_{\underline{w}}^{\overline{w}} \tilde{z}_\infty(w)\, g(w)\, \mathrm{d}w \leq 1$, and is as follows (Tonis 1998):

$$\tilde{z}_\infty(w) = F\left(\frac{[\rho(w)]_+}{A_\infty}\right), \; \tilde{s}_\infty(w) = w\varphi(\tilde{z}_\infty(w)) - \int_{\underline{w}}^{w} \varphi(\tilde{z}_\infty(t))\, \mathrm{d}t, \tag{26}$$

where $A = A_\infty$ is the unique solution of the equation

$$\int_{\underline{w}}^{\overline{w}} F\left(\frac{[\rho(w)]_+}{A}\right) g(w)\, \mathrm{d}w = 1 \tag{27}$$

(it is assumed through the end of this section that $\int_{\underline{w}}^{\overline{w}} F\left(\frac{[\rho(w)]_+}{A}\right) g(w)\, dw < \infty$, $\forall A > 0$). The function $\tilde{z}_\infty(w)$, and hence $\tilde{s}_\infty(w)$, are monotonically increasing for $w \in [w^0, \overline{w}]$, and

$$z_\infty(s) \equiv \tilde{z}_\infty\left(\tilde{s}_\infty^{-1}(s)\right), \quad s \in [\underline{s}_\infty, \overline{s}_\infty], \tag{28}$$

with $\underline{s}_\infty = \tilde{s}_\infty\left(\underline{w}\right), \overline{s}_\infty = \tilde{s}_\infty(\overline{w})$, is a rent-seeking success function, which in the present "atomless" case depends only on an agent's own contribution. We will now show that this function approximates optimal CSFs (14) when the number of participating agents is large.

To this end, suppose, as in Mas-Colell and Vives (1993), that n agents with types $w_1, \ldots, w_n$ are randomly and independently drawn from the distribution $G(w)$ to obtain a discrete approximation of the said distribution, so that each agents carries a weight $1/n$. This means that if z_i and s_i are resp. the resource allocated to agent i and his payment, then the resource constraint takes form $\sum_{i=1}^n \frac{1}{n} z_i \leq 1$, and similarly the resource administrator's revenue equals $\sum_{i=1}^n \frac{1}{n} s_i$. The optimal CSF-based mechanism for such sample $z_i^{(n)}(s_i, s_{-i})$ with $s_i \in \left[\underline{s}^{(n)}, \overline{s}^{(n)}\right]$ is described above (superscript n stands for the size of the sample) with the only modification that $A_F^{(n)}$ now satisfies the following equation:

[17] Similar setups are used in the optimal taxation theory and in the more general literature on mechanism design with continuum of agents (see e.g., Mas-Colell and Vives 1993).

$$\sum_{i=1}^{n} \frac{1}{n} F\left(\frac{x_i}{A_F^{(n)}\left(x_1, \ldots, x_n\right)}\right) = 1. \tag{11'}$$

Functions (28) approximate CSFs (14), (11′) "on the average" in the following sense: when contributions of all agents but i are fixed at their equilibrium levels $s_j = \tilde{s}^{(n)}\left(w_j\right)$, $j = 1, \ldots, n$; $j \neq i$, one obtains a parametric family of single-variable contest success functions $z_i^{(n)}\left(s_i | w_{-i}\right) \equiv z_i^{(n)}\left(s_i, \tilde{s}_{-i}^{(n)}\left(w_{-i}\right)\right)$, and according to the following proposition, for a given outlay s_i the expected value of such functions over other agents' types approaches $z_\infty\left(s_i\right)$ for large n.[18] We establish such convergence in the next two propositions under an additional technical assumption $\rho\left(\underline{w}\right) > 0$.

Proposition 6 *Domains of CSFs* $z_i^{(n)}\left(s_i, s_{-i}\right)$ *approximate those of* $z_\infty(s)$*:* $\lim_{n\to\infty} \underline{s}^{(n)} = \underline{s}_\infty$, $\lim_{n\to\infty} \overline{s}^{(n)} = \overline{s}_\infty$, *and*

$$\lim_{n\to\infty} E_{w_{-i}}\, z_i^{(n)}\left(s_i \mid w_{-i}\right) = z_\infty\left(s_i\right), \quad \forall s_i \in \left(\underline{s}, \overline{s}\right).^{19} \tag{29}$$

For Cobb-Douglas utilities, when according to Proposition 4 optimal CSFs allow a logit representation

$$z_i^{(n)}\left(s_1, \ldots, s_n\right) = \frac{\xi^{(n)}(s_i)}{\frac{1}{n}\sum_{j=1}^{n} \xi^{(n)}(s_j)}, \tag{30}$$

where functions $\xi^{(n)}(s) = \left(\eta^{(n)}(s)\right)^{\frac{1}{1-\alpha}}$ and $\eta^{(n)}(\cdot)$ are calculated according to (16), Proposition 6 can be re-stated as convergence of $\xi^{(n)}$ to a constant multiple of $z_\infty(\cdot)$.

Proposition 7 *If* $\varphi(z) = Cz^\alpha$, $\alpha \in (0, 1)$, *then for* $\xi^{(n)}\left(s_i\right) = \left[\rho\left(\left(\tilde{s}^{(n)}\right)^{-1}(s)\right)\right]^{\frac{1}{1-\alpha}}$ *one has*

$$\begin{aligned}\lim_{n\to\infty} \xi^{(n)}(s_i) &= \xi_\infty\left(s_i\right) \equiv \left[\rho\left(\tilde{s}_\infty^{-1}\left(s_i\right)\right)\right]^{\frac{1}{1-\alpha}} \\ &= z_\infty\left(s_i\right) \int_{\underline{w}}^{\overline{w}} \left[\rho(w)\right]^{\frac{1}{1-\alpha}} g(w)\, \mathrm{d}w, \quad \forall s_i \in \left(\underline{s}, \overline{s}\right).\end{aligned} \tag{31}$$

[18] In exchange economies with a continuum of agents a weaker convergence result holds: any accumulation point of Bayes-Nash equilibria in approximating finite economies is an equilibrium in the limiting continuum economy (Mas-Colell and Vives 1993).

[19] It is assumed that n is large enough to have $s_i \in \left[\underline{s}^{(n)}, \overline{s}^{(n)}\right]$.

Properties of the limit CSF z_∞ can now be extended in the above described sense on the optimal CSFs $z_i^{(n)}$, when the number of agents is sufficiently large. One such property is increasing returns to scale which holds under a mild additional assumption.[20]

Proposition 8 *If the ratio $\rho(w)/w$ monotonically non-decreases, then the limiting CSF $z_\infty(s_i)$ is convex.*[21]

According to the above proposition, when agents are sufficiently numerous, those among them with higher valuation of the source of rent (larger endowments of a complementary input) obtain the resource allocated by the administrator on increasingly better terms (whereas, as it was noted above, agents at the bottom of the type distribution could even opt out of rent seeking altogether).[22] Such discrimination leads to re-distribution of the allocated resource (in comparison with the socially optimal competitive benchmark when the resource is sold at the market-clearing price) from "low" to "high" types to whom optimal CSFs give a scale advantage.[23]

Consider as an example the uniform distribution of w on the [2, 3] range and agents' utility functions with $\varphi(z) = 2\sqrt{z}$. In this case, the limiting function $\xi_\infty(s_i) = 12 + 4.16\, s_i - 8\sqrt{2 + 1.04 s_i}$, defined over the [1.92, 6.73] range; the graph of this function and convergence to it of $\xi^{(n)}(s_i)$ are shown on Fig. 2.

Another noteworthy example can be obtained by combining the Pareto distribution $G(w) = 1 - (\underline{w}/w)^k$ $(\underline{w} \le w < \infty)$ and Cobb-Douglas utility $\varphi(z) = \alpha^{-1} z^\alpha$ with $\alpha \in (0, 1)$, $k(1-\alpha) > 1$. In this case, the limit of the functions $\xi^{(n)}(s_i)$ entering optimal CSFs (2) for finite n is as follows:

$$\xi_\infty(s_i) = C(s_i/\underline{s} - (1-\alpha)),$$

where $\underline{s} = \underline{w}\left(1 - \frac{1}{k(1-\alpha)}\right)^\alpha/\alpha$, and $C = \frac{1}{\alpha}\left[\underline{w}\left(1 - \frac{1}{k}\right)\right]^{1/1-\alpha}$. Here the limiting function ξ_∞ is linear in agents' outlays, similarly to Tullock's initial fractional model (1).

6 An extension: variable resource

It was assumed so far that the stock of resource allocated by the administrator is fixed; however, in many applications it can be expanded at some additional cost to the administrator. To explore such situations, we assume in this section that the administrator has the option to partially invest payments collected from rent-seeker to augment the allocated resource. Such modification leads to a yet another class of CSFs.

[20] Increasing returns to scale in rent-seeking activities were posited in a different context in Murphy et al. (1993) and studied by Cornes and Hartley (2005). More generally on the role of economy of scale in rent-seeking, see Tullock (1980).

[21] This condition holds if e.g., the hazard rate $\frac{g(w)}{1-G(w)}$ of the distribution $G(w)$ monotonically increases.

[22] Such feature commonly occurs in optimal contracts due to the single-crossing property.

[23] For interpretation of optimal auctions as monopolistic price discrimination see Bulow and Roberts (1989).

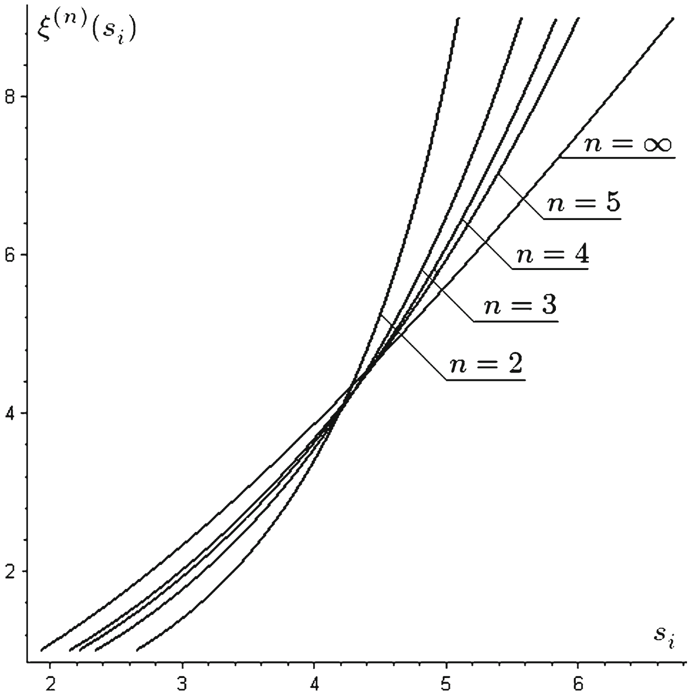

Fig. 2 Convergence of CSFs

Namely, suppose that the administrator has access to a resource-production technology with monotonically increasing and concave production function $\mathcal{F}(s)$. If the administrator invests in this technology an amount s_0 from her total receipts and keeps the balance $\sum_{i=1}^{n} s_i - s_0$, then she will have $\mathcal{F}(s_0)$ units of the resource available for allocation to rent-seekers. To obtain endogenous CSFs in this setting, the procedure presented in Sect. 4 is still applicable, with the following adjustment: direct mechanisms now include, in addition to functions $\tilde{s}_i$ and $\tilde{z}_i$, a yet another function $\tilde{s}_0(w_1, \dots w_n)$, which together satisfy the constraints

$$\sum_{i=1}^{n} \tilde{z}_i(w_i,\ w_{-i}) \le \mathcal{F}(\tilde{s}_0(w_1, \dots w_n)); \quad \tilde{s}_0(w_1, \dots w_n) \le \sum_{i=1}^{n} \tilde{s}_i(w_i). \tag{32}$$

The optimal direct mechanism maximizes the expected payoff of the administrator

$$\max E\left[\sum_{i=1}^{n} \tilde{s}_i\ (w_i) - \tilde{s}_0\ (w_1, \dots w_n)\right] \tag{33}$$

subject to constraints (5′), (6′) and (32). The rest of the procedure remains the same, and its outcome, assuming again agents' Cobb-Douglas utilities $\varphi\ (z) = \alpha^{-1} z^{\alpha}$, is as follows.

Proposition 9 *Optimal CSFs admit representation*

$$z_i(s_i, s_{-i}) = H\left(\sum_{j=1}^{n} \xi(s_j)\right) \frac{\xi(s_i)}{\sum_{j=1}^{n} \xi(s_j)} \tag{34}$$

with some monotonically increasing functions ξ *and* H*. Here* $H(t) \equiv \mathcal{F}(\Psi(t))$*, where* $\Psi(\cdot)$ *is an inverse function to* $\mathcal{F}(s)/[\mathcal{F}'(s)]^{1/1-\alpha}$*;* $\xi(s) = (\eta(s))^{\frac{1}{1-\alpha}}$*, and functions* $\eta(s)$ *are calculated according to* (12), (13) *with the underlying direct mechanism*

$$\tilde{z}_i(w_i, w_{-i}) = H\left(\sum_{j=1}^{n} \rho(w_j)^{1/1-\alpha}\right) \frac{\rho(w_i)^{1/1-\alpha}}{\sum_{j=1}^{n} \rho(w_j)^{1/(1-\alpha)}}. \tag{35}$$

Endogenous CSFs combine features of the functional forms (2) and (3); e.g., for a Cobb-Douglas resource production technology $\mathcal{F}(s) = s^{\beta}$, $0 < \beta < 1$, one obtains $H(t) = Ct^{(\beta-\alpha\beta)/(1-\alpha\beta)}$, for some $C > 0$. Notice that according to (34) the rent-seeking contest acquires features of cooperative production, since rent-seekers' contributions also increase the total supply of resource.

This mechanism is also not ex post efficient, and variable resource entails additional efficiency losses, on top of those observed in Sect. 4: the equilibrium investment falls short of the ex post social optimum. Indeed, it follows from (35) that $\tilde{s}_0(w_1, \ldots w_n) = \Psi\left(\sum_{j=1}^{n} \rho(w_j)^{1/1-\alpha}\right)$, whereas it is easy to verify that the first-best investment s^* is as follows: $s^*(w_1, \ldots w_n) = \Psi\left(\sum_{j=1}^{n} w_j{}^{1/1-\alpha}\right)$, and since Ψ monotonically increases (as an inverse to a monotonically increasing function), and $\rho(w) < w$, $\forall w < \overline{w}$, one has $\tilde{s}_0(w_1, \ldots w_n) < s^*(w_1, \ldots w_n)$, unless all agents are of the highest possible type. Such efficiency losses are due to the administrator's inability to fully appropriate the resource rent which is partly shared with rent-seeking agents—full rent appropriation is precluded by informational asymmetry.[24] Notice that efficiency losses which are conventionally associated with the "tragedy of the commons" are due to overinvestment in cooperative production (see e.g., Moulin and Watts 1997), whereas in the present case inefficiency is caused by *under*investment.

7 Concluding remarks

The paper contributes to the strand of public choice literature where rules of rent-seeking contests are not assumed upfront but instead are endogenous to some behavioral, institutional, informational, etc. assumptions. Our analysis summarizes as follows. First, we show that in the case of independent private values optimal mechanisms can always be implemented via some CSFs $z_i(s_1, \ldots, s_n)$ as initially posited by Tullock. Second, such optimal endogenous CSFs are shown to have properties

[24] Similarly in McGuire and Olson (1996), an autocratic regime under-invests in its tax base (in comparison to the social optimum) due to deadweight losses of taxation.

which are commonly assumed a priori as plausible features of rent-seeking contests. The paper, therefore, validates such assumptions in the situations where contest rules are set by an administrator under informational asymmetry. Third, we identify conditions when optimal mechanisms entail logit CSFs, and similarly, in the case of variable resource, cooperative production-type CSFs. Fourth, we offer a simple unidimensional approximation of optimal CSFs for large numbers of participating agents.

Analysis of endogenous CSFs sheds light on a number of distributional issues of public choice and political economy, such as discrimination of small stake holders and increasing returns in rent seeking. It reveals origins of ex post efficiency losses in rent seeking, including the failure to achieve socially optimal investments into rent-generating resources. Such losses are in fact preventable, since in the setup considered in the paper there always exist ex post efficient incentive-compatible Groves mechanisms satisfying all feasibility constraints, but those mechanisms will not be selected by a revenue-maximizing resource administrator.

The above analysis can be extended in several ways, to reflect variations in the setups of rent-seeking and auction theory (Klemperer 1999; Corchón 2007; Congleton et al. 2008). Such extensions include, but are not limited to, bidders' asymmetry; risk-aversion; collective rent seeking and possibility of collusion; more complex preferences of the administrator that combine private and public interest; entry costs; etc., and are left to future research.

Appendix

Proof of Proposition 1 The problem (5′)–(8′) is solved by using tools of the mechanism design/optimal auction theory (Myerson 1981; Klemperer 1999; for the case of divisible prize see also Maskin and Riley 1989). Consider agents' expected net equilibrium payoffs

$$\pi_i\left(w_i\right) \equiv w_i\; E_{w_{-i}}\varphi\left(\tilde{z}_i\left(w_i, w_{-i}\right)\right) - \tilde{s}_i\left(w_i\right), \quad i = 1, \ldots, n. \tag{A.1}$$

Assuming interior w_i and differentiability, the necessary condition for incentive compatibility (5′) is

$$w_i \overline{\varphi}_i'\left(w_i\right) = \tilde{s}_i'\left(w_i\right), \tag{A.2}$$

where

$$\overline{\varphi}\left(w_i\right) \equiv E_{w_{-i}}\varphi\left(\tilde{z}_i\left(w_i, w_{-i}\right)\right), \tag{A.3}$$

or equivalently,

$$\pi_i'\left(w_i\right) \equiv \overline{\varphi}_i\left(w_i\right), \tag{A.4}$$

for all $i = 1, \ldots, n$. Furthermore, incentive compatibility constraints are satisfied if and only if equalities (A.4) hold and functions $\overline{\varphi}_i\left(w_i\right)$ are monotonically non-decreasing.

According to (A.4), functions π_i are non-decreasing, and therefore once the participation constraint (6′) is satisfied for the lowest type $\underline{w}$, it holds for all other types. In the optimum $\pi_i(0) = 0$, so that $\pi_i(w_i) = \int_{\underline{w}}^{w_i} \overline{\varphi}_i(s)\, \mathrm{d}s$ and hence

$$\tilde{s}_i(w_i) = w_i \overline{\varphi}_i(w_i) - \int\limits_{\underline{w}}^{w_i} \overline{\varphi}(s)\, \mathrm{d}s. \tag{A.5}$$

Substituting (A.5) into the administrator's objective function, one obtains

$$\begin{aligned}\int\limits_{\underline{w}}^{\overline{w}} \tilde{s}_i(w_i)\, g(w_i)\, \mathrm{d}w_i &= \int\limits_{\underline{w}}^{\overline{w}} w_i\, \overline{\varphi}_i(w_i)\, g(w_i)\, \mathrm{d}w_i - \int\limits_{\underline{w}}^{\overline{w}} \int\limits_{\underline{w}}^{w_i} \overline{\varphi}_i(t)\, \mathrm{d}t\, g(w_i)\, \mathrm{d}w_i \\ &= \int\limits_{\underline{w}}^{\overline{w}} \left[w_i - \frac{1 - G(w_i)}{g(w_i)} \right] \overline{\varphi}_i(w_i)\; g(w_i)\, \mathrm{d}w_i.\end{aligned}$$

Administrator's gross payoff can thus be represented as

$$\int\limits_{\underline{w}}^{\overline{w}} \ldots \int\limits_{\underline{w}}^{\overline{w}} \left[\sum_{i=1}^{n} \rho(w_i)\, \varphi(\tilde{z}_i(w_i, w_{-i})) \right] g(w_1) \ldots g(w_n)\, \mathrm{d}w_1 \ldots\; \mathrm{d}w_n,$$

and, ignoring for a moment constraints (5′), (6′), functions $\tilde{z}_i(\cdot)$ can be found from the following problems:

$$\begin{aligned}&\max \left[\sum_{i=1}^{n} \rho(w_i)\, \varphi(\tilde{z}_i(w_i, w_{-i})) \right], \sum_{i=1}^{n} \tilde{z}_i(w_i,\; w_{-i}) \leq 1,\; \tilde{z}_j\left(w_j, w_{-j}\right) \geq 0, \\ &j = 1, \ldots, n,\end{aligned} \tag{A.6}$$

for any $w_j \in \left[\underline{w}, \overline{w}\right],\; j = 1, \ldots, n$. This is a standard resource allocation problem, and given the neo-classical properties of φ, its solution is as follows:

$$\begin{aligned}&\rho(w_i)\, \varphi'(\tilde{z}_i(w_i, w_{-i})) = \lambda(w_1, \ldots, w_n), \quad \text{for all } i = 1, \ldots, n \text{ such that} \\ &\quad \rho(w_i) > 0;\, \tilde{z}_i(w_i, w_{-i}) = 0, \quad \text{for all } i = 1, \ldots, n \text{ such that } \rho(w_i) \leq 0.\end{aligned}$$

Solving for $\lambda(w_1, \ldots, w_n)$ from the budget constraint $\sum_{i=1}^{n} \tilde{z}_i(w_i, w_{-i}) \leq 1$, one obtains (11); the Eq. (10) indeed has a unique solution, since F is monotonically increasing and $F(0) = 0,\; F(t) \to \infty$ with $t \to \infty$. The mechanism is made complete by combining $\tilde{z}_i$ with agents' contribution functions $\tilde{s}_i$ derived according to (A.3), (A5); notice that solution (11) is symmetric, and hence the subscript i in $\overline{\varphi}_i$ can be dropped.

To verify optimality, notice that if $\mu_i > 0$ and at least for some $j \neq i$ $\mu_j > 0$, then in the optimal solution of the problem

$$\max \sum_{k=1}^{n} \mu_k \varphi(z_k), \quad \sum_{k=1}^{n} z \leq 1, \quad z_l \geq 0, \quad l = 1, \ldots, n, \tag{A.7}$$

x_i monotonically increases in μ_i.[25] Therefore, $\tilde{z}_i(w_i, w_{-i})$ monotonically increases in w_i over the range $w_i \in [w^0, \overline{w}]$ if at least some other w_j is greater than w^0, and monotonically non-decreases (being equal to zero) otherwise. This means that the expected value $\overline{\varphi}(w_i) \equiv E_{w_{-i}} \varphi(\tilde{z}_i(w_i, w_{-i}))$ monotonically increases in $w_i \in [w^0, \overline{w}]$, since $w_j > w^0$ with positive probability. Therefore, the obtained mechanism indeed maximizes (7′) subject to (5′), (6′), and (8′): participation constraint is met by definition, whereas incentive compatibility follows from (A.4) and monotonicity of $\overline{\varphi}$. □

Proof of Proposition 2 Monotonicity of $\overline{\varphi}$ implies that the function $\tilde{s}(w_i)$ monotonically increases over the same range $[w^0, \overline{w}]$; indeed if $x < y$, $x, y \in [w^0, \overline{w}]$, then $\tilde{s}(y) - \tilde{s}(x) = (y - x)\overline{\varphi}(y) + x(\overline{\varphi}(y) - \overline{\varphi}(x)) - \int_x^y \overline{\varphi}(t)\,dt > x(\overline{\varphi}(y) - \overline{\varphi}(x)) > 0$. This allows to invert $\tilde{s}$ and define CSFs (14). These CSFs deliver (as a Bayes-Nash equilibrium with agents' strategies $s_i(w_i) = \tilde{s}(w_i)$, $i = 1, \ldots, n$) the same outcomes as the optimal direct mechanism (11), (12), and participation constraint (6) follows from (6′). □

Proof of Proposition 3 Symmetry, monotonicity, and constant returns to scale of A_F follow immediately from its definition. Since F is monotonically increasing, one has

$$1 = \sum_{i=1}^{n} F\left(\frac{x_i}{A_F(x_1, \ldots, x_n)}\right) \leq nF\left(\frac{\max x_i}{A_F(x_1, \ldots, x_n)}\right),$$

and therefore $\frac{\max x_i}{A_F(x_1,\ldots,x_n)} \geq t_n$; the inequality for $\min x_i$ is established similarly. Monotonicity of A_F implies that $F\left(\frac{x_i}{A_F(x_1,\ldots,x_n)}\right)$ monotonically decreases in x_j for $j \neq i$, and due to the constraint $1 = \sum_{i=1}^{n} F\left(\frac{x_i}{A_F(x_1,\ldots,x_n)}\right)$, monotonically increases in x_i. Hence $\tilde{z}_i(w_i, w_{-i})$ monotonically increases in w_i over the range $w_i \in [w^0, \overline{w}]$ (which has been already established in the proof of Proposition 1) and monotonically decreases in w_j, $j \neq i$ over the same range. To complete the proof, notice that the function $\eta(s) \equiv \rho(\tilde{s}^{-1}(s))$ monotonically increases, since the marginal valuation function $\rho(\cdot)$ increases by assumption, and $\tilde{s}(\cdot)$—due to Proposition 2. □

Proof of Proposition 4 Only the second part of the proposition needs to be verified. Let $\underline{x} \equiv \eta(\underline{s})$, $\overline{x} \equiv \eta(\overline{s})$, and for all $x_1, \ldots, x_n \in [\underline{x}, \overline{x}]$ and some monotonically

[25] Re-write (A.7) as $\max \mu_i \varphi(z_i) + \Phi(z_i)$, $0 \leq z_i \leq 1$, where $\Phi(t) \equiv \max \sum_{k \neq i} \mu_k \varphi(z_k)$, $\sum_{k \neq i} z_k \leq 1 - t$, $z_l \geq 0$, $l \neq i$.

increasing function $\zeta(\cdot)$ one has

$$F\left(\frac{x_i}{A_F(x_1,\ldots,x_n)}\right)=\frac{\zeta(x_i)}{\sum_{j=1}^{n}\zeta(x_j)},\quad i=1,\ldots,n. \tag{A.8}$$

Denote $y_i=\zeta(x_i)$, $\mu(\cdot)\equiv\zeta^{-1}$, in which case (A.8) yields

$$\frac{\mu(y_1)}{\mu(y_2)}=\frac{F^{-1}\left(\frac{y_1}{\sum_{i=1}^{n}y_i}\right)}{F^{-1}\left(\frac{y_2}{\sum_{i=1}^{n}y_i}\right)}.$$

Let $y,t>0$ be such that $y,ty\in\left[\zeta\left(\underline{x}\right),\zeta\left(\overline{x}\right)\right]$; by letting $y_1=ty$, $y_i=y$, $i=2,\ldots,n$, one obtains from the previous equality

$$\mu(ty)=\mu(y)\,\frac{F^{-1}\left(\frac{t}{t+(n-1)}\right)}{F^{-1}\left(\frac{1}{t+(n-1)}\right)}. \tag{A.9}$$

Differentiating (A.9) with respect to y (the function μ is smooth due to (A.8)) yields $t\mu'(ty)=\mu'(y)\,\mu(ty)/\mu(y)$, and therefore

$$\frac{ty\mu'(ty)}{\mu(ty)}=\frac{y\mu'(y)}{\mu(y)},$$

for all y,t. Therefore, the function μ has constant elasticity over its domain, and hence $\mu(y)=C_1y^{\sigma}$, for some $C_1,\sigma>0$; consequently $\zeta(x)=C_2x^{1/\sigma}$, $C_2>0$.

One can easily check that for any $x\in[\underline{x},\overline{x}]$ there exists unique $\tilde{x}\in[\underline{x},\overline{x}]$ such that $A_F\left(x,\underline{x},\tilde{x},\ldots,\tilde{x}\right)=A_F\left(\underline{x},\underline{x},\overline{x},\ldots,\overline{x}\right)\equiv A_0$ (recall that $n>2$). Indeed, $A_F\left(x,\underline{x},\underline{x},\ldots,\underline{x}\right)\leq A_0\leq A_F\left(x,\underline{x},\overline{x},\ldots,\overline{x}\right)$. Hence, due to (A.8),

$$\frac{F\left(\frac{x}{A_0}\right)}{F\left(\frac{\underline{x}}{A_0}\right)}=\frac{F\left(\frac{x}{A_F(x,\underline{x},\tilde{x},\ldots,\tilde{x})}\right)}{F\left(\frac{\underline{x}}{A_F(x,\underline{x},\tilde{x},\ldots,\tilde{x})}\right)}=\frac{x^{1/\sigma}}{\underline{x}^{1/\sigma}},$$

which leads to $F(t)=C_3t^{1/\sigma}$, $C_3>0$. Finally, due to $F(t)\equiv\left(\varphi'\right)^{-1}(1/t)$ one obtains $\varphi'(z)=C_4z^{-\sigma}$, $C_4>0$. Integration leads to $\varphi(z)=B_0+B_1z^{\alpha}$, where due to φ's monotonicity and strict concavity $\alpha\equiv1-\sigma\in(0,1)$ and $B_1>0$. If $\underline{z}=0$, then $\varphi(0)=0$ implies $B_0=0$; otherwise to allow concave extrapolation of φ on $[0,\underline{z}]$, one should have $\frac{\varphi(\underline{z})}{\underline{z}}>\varphi'(\underline{z})$, or $B_0+(1-\alpha)\,\underline{z}^{\alpha}B_1>0$. □

Proof of Proposition 5 According to the equivalence result (Williams 1999), when agents' utilities are quasi-linear and their types are distributed independently, any ex post efficient and incentive-compatible mechanism generates the same expected utilities for participating agents as an appropriately chosen Groves mechanism (23).

Therefore, it suffices to show that constants $k_1, \ldots, k_n$ can be chosen to satisfy the participation constraints (6′) and (22).[26]

For the Groves mechanisms (23), contestants' expected utilities

$$E_{w_{-i}} \sum_{j=1}^{n} w_j \varphi\left(\tilde{z}_j(w)\right) - k_i$$

are at their lowest levels when $w_i = \underline{w}$, and therefore participation constraints (6′) are satisfied if and only if

$$E_{w_{-i}} \left[\underline{w} \varphi\left(\tilde{z}_i\left(\underline{w}, w_{-i}\right)\right) + \sum_{j \neq i} w_j \varphi\left(\tilde{z}_j\left(\underline{w}, w_{-i}\right)\right) \right] \geq k_i, \quad i = 1, \ldots, n,$$

and

$$E\left[\sum_{i=1}^{n} \sum_{j \neq i} w_j \varphi\left(\tilde{z}_j(w)\right)\right] \leq \sum_{i=1}^{n} k_i.$$

The above inequalities are jointly satisfied for some $k_1, \ldots, k_n$ if and only if

$$(n-1) E\left[\sum_{i=1}^{n} w_i \varphi(\tilde{z}_i(w))\right] \leq n E\left[\sum_{i=1}^{n-1} w_i \varphi\left(\tilde{z}_i\left(\underline{w}, w_{-n}\right)\right) + \underline{w} \varphi\left(\tilde{z}_n\left(\underline{w}, w_{-n}\right)\right)\right] \tag{A.10}$$

Notice that the LHS of (A.10) equals $n(n-1)B$ with $B \equiv E\left[w_1 \varphi(\tilde{z}_1(w))\right]$, whereas the RHS of the same inequality is not less than $n(n-1)C$ where $C \equiv E\left[w_1 \varphi(\tilde{z}_1\left(\underline{w}, w_{-n}\right))\right]$. Recall that $\tilde{z}_1(w)$ monotonically non-increases in w_n, and therefore $C \geq B$, which concludes the proof.[27] □

Proof of Proposition 6 Fix w_i and treat w_j, $j \neq i$ as independent random variables. According to the law of large numbers (Feller 1968), for every given $A > 0$ the random variable $\sum_{k=1}^{n} \frac{1}{n} F\left(\frac{\rho(w_k)}{A}\right)$ converges in probability to $EF\left(\frac{\rho(w)}{A}\right) \int_{\underline{w}}^{\overline{w}} F\left(\frac{\rho(w)}{A}\right) g(w)\, dw$. This implies that $A_F^{(n)}(\rho(w_1), \ldots, \rho(w_n))$ converges in probability to A_∞ (recall that F monotonically increases), and hence

[26] The administrator is treated as $n+1$th agent who derives utility solely from transfer payments. Alternately only contestants could be considered as agents, in which case the inequality (22) becomes a budget balance condition.

[27] The existence of an ex post efficient mechanism with all other desired properties is owed to the fact that expected aggregate gains from optimal resource allocation are sufficiently high even if one of the agents is in the least advantageous position ($w_i = \underline{w}$). This leaves room for "taxes" $k_1, \ldots, k_n$ which are high enough to meet the budget balance condition (22) without breaking the participation constraints. In contrast, in the well-known case of bilateral trade such mechanism does not exist (Myerson and Satterthwaite 1983) because worst-off traders cannot realize any gains from trade (Williams 1999).

$\tilde{z}_i^{(n)}(w_i, w_{-i}) = F\left(\frac{\rho(w_i)}{A_F^{(n)}(\rho(w_1), ..., \rho(w_n))}\right)$ converges in probability to $F\left(\frac{\rho(w_i)}{A_\infty}\right) = \tilde{z}_\infty(w_i)$. Notice that $A_F^{(n)}(\rho(w_1), \ldots, \rho(w_n)) \leq A_{\max}$ for all n, $w_1, \ldots, w_n$, where $F\left(\frac{\rho(w)}{A_{\max}}\right)$=1, and so random variables $\tilde{z}_i^{(n)}(w_i, w_{-i})$ are bounded from above by $F\left(\frac{\rho(w_i)}{A_{\max}}\right)$; therefore, convergence of these variables in probability implies convergence of their expected values. Therefore,

$$\lim_{n\to\infty} E_{w_{-i}}\tilde{z}_i^{(n)}(w_i, w_{-i}) = \tilde{z}_\infty(w_i). \tag{A.11}$$

By the same token $\overline{\varphi}^{(n)}(w_i) \equiv E_{w_{-i}}\varphi\left(\tilde{z}_i^{(n)}(w_i, w_{-i})\right) \to \varphi(\tilde{z}_\infty(w_i))$, $n \to \infty$, and hence

$$\lim_{n\to\infty} \tilde{s}^{(n)}(w_i) = \tilde{s}_\infty(w_i). \tag{A.12}$$

Letting in (A.12) w_i equal $\underline{w}$ and $\overline{w}$, one obtains respectively $\lim_{n\to\infty} \underline{s}^{(n)} = \underline{s}_\infty$, $\lim_{n\to\infty} \overline{s}^{(n)} = \overline{s}_\infty$. The functions $\tilde{s}^{(n)}(w_i)$ are monotonically increasing, and therefore due to (A.12) the inverses of these functions converge to $\tilde{s}_\infty^{-1}(\cdot)$. This fact in combination with (A.11) and the observation that functions $E_{w_{-i}}\tilde{z}_i^{(n)}(\cdot, w_{-i})$ are also monotonically increasing, leads to (29). □

Proof of Proposition 7 It was shown in the proof of Proposition 6 that functions $\left(\tilde{s}^{(n)}\right)^{-1}(\cdot)$ converge to $\tilde{s}_\infty^{-1}(\cdot)$. □

Proof of Proposition 8 The first-order version of the incentive compatibility condition (24) implies that $\frac{dz_\infty}{ds} = \frac{d\tilde{z}_\infty}{dw} / \frac{d\tilde{s}_\infty}{dw} = \frac{1}{w\varphi'(\tilde{z}_\infty(w))}$. One also has $\rho(w)\varphi'(\tilde{z}_\infty(w)) = A_\infty$, and hence $\frac{dz_\infty}{ds} = \frac{\rho(w)}{A_\infty w}$ non-decreases in w. Finally, $\tilde{s}_\infty(w)$ monotonically increases in w, and hence $\frac{dz_\infty}{ds}$ monotonically non-decreases in s. □

Proof of Proposition 9 As in the proof of Proposition 1, optimal direct mechanism design boils down to the following problem similar to (A.6):

$$\max\left[\sum_{i=1}^{n} \rho(w_i)\varphi(\tilde{z}_i(w_i, w_{-i})) - \tilde{s}_0(w_1, \ldots, w_n)\right]$$

$$\sum_{i=1}^{n} \tilde{z}_i(w_i, w_{-i}) \leq \mathcal{F}(s_0(w_1, \ldots, w_n)), \tilde{z}_j\left(w_j, w_{-j}\right) \geq 0, j = 1, \ldots, n. \tag{A.13}$$

Assuming an interior optimum, one has

$$\tilde{z}_i(w_i, w_{-i}) = \rho(w_i)^{1/1-\alpha}(\mathcal{F}'(\tilde{s}_0(w_1, \ldots w_n))^{1/1-\alpha}, \tag{A.14}$$

and due to the budget constraint of the problem (A.13),

$$\mathcal{F}(\tilde{s}_0\,(w_1,\ldots w_n) = (\mathcal{F}'(\tilde{s}_0(w_1,\ldots w_n))^{1/1-\alpha}\sum_{i=1}^{n}\rho(w_i)^{1/1-\alpha} \qquad \text{(A.15)}$$

(A.14) and (A.15) yield (35). Similarly to the proof of Proposition 1 it can be shown that here too $\tilde{z}_i\,(w_i, w_{-i})$ monotonically increases in $w_i \in \left[w^0, \overline{w}\right]$ if at least for some other agents $w_j > w^0$, and monotonically non-decreases (being equal to zero) otherwise, and therefore the allocation (35) is indeed a part of optimal direct mechanism. Arguments similar to those presented in the proof of Proposition 2 complete the proof of Proposition 9. □

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